Math Capstone: The Nerve of a Category

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1 Introduction

We give an intuitive description of the the nerve of a category, assuming only basic understanding of category theory, as well an understanding of point-set topology equivalent to one undergraduate course in the subject, including a basic understanding of CW complexes. As the nerve of a category may be defined in terms of simplicial sets, we first give a detailed description of simplicial sets and some of the prerequisite concepts required to understand them. We then explain the concept of the nerve in detail with some simple examples, before showing that (the geometric realization of) the nerve of the cyclic group of order 2 (C_2) represented as a category is homeomorphic to infinite-dimensional real projective space ($\mathbb{RP}^\infty$).

2 An introduction to simplicial sets

2.1 Simplices and (Abstract) Simplicial Complexes

An *n-simplex* can be thought of as a generalized triangle: a 0-simplex is a point, a 1-simplex is a line, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. Formally, a geometric n-simplex is a convex set that is spanned by $n+1$ geometrically independent points (called *vertices*) in euclidean space. A face of an n-simplex is a convex set spanned by a subset of these vertices.[2] These simplexes are uniquely determined by their vertices, so we can write a simplex as a simple array of the vertices, e.g $[v_0, v_1, v_2]$.

Definition 2.1. A **simplicial complex** X is a collection of simplices in $\mathbb{R}^N$ such that

1. every face of every simplex in X is in turn in X
2. if two simplexes in X intersect, their intersection is a face of both simplexes.

This s a very geometric definition, but oftentimes we do not care about the particular realization in euclidean space, which leads us to the idea of an *abstract simplicial complex*, where we consider the vertices merely combinatorically, as abstract points rather than particular points in euclidian space.

Definition 2.2. An **Abstract Simplicial Complex** $\mathcal{S}$ is a collection of finite, non-empty subsets, such that if A is an element of $\mathcal{S}$, then so is every non-empty subset of A.[3]

Note the correspondence to the geometric version — if we interpret each set with n elements as a n-simplex, then the singleton sets are the vertices, and the requirement that all non-empty subsets of each set is in the collection corresponds with the intuitive notion that every face of each simplex is in the complex.

We can make this correspondence more precise by considering *the geometric realization* of an abstract simplicial complex. We say that a geometric simplicial complex S is the geometric realization of an abstract simplicial complex S' if the set $\mathcal{K}$ obtained by including every set of vertices that span a simplex in S is isomorphic to S'.

A key idea for later abstractions is that of the **simplicial map**. Speaking loosely, there can be **inclusion maps**, where a simplex is mapped into a larger simplicial complex that contains it, and **face maps** where a simplex is mapped to one of its faces. Face maps are maps that take an n-simplex to an

$(n-1)$-simplex by simply eliding one of the vertices. We notate the face maps as d_i, where i is the index of the vertex that will be removed, such that, for example, $d_1([v_0, v_1, v_2]) = [v_0, v_2]$.

Note that here we are assigning an implicit order to the elements in order to easily refer to which vertex is being removed. We can take this a step further to for a bit of of notational convenience and refer to the *nth* vertex just as n, i.e such that we can have a simplex $[0, 1, 2, 3]$.

2.2 Delta Sets and Simplicial Sets

We can use this notion of face map to generalize simplicial complexes and define **Delta Sets**, an intermediate abstraction between simplicial sets and simplicial complexes.

Definition 2.3. A **Delta set** is a sequence of sets $X_0, X_1, \ldots$ such that for each $n \geq 0$, we have maps $d_i : X_{n+1} \to X_n$ for $0 \leq i \leq n+1$ such that $d_i d_j = d_{j-1} d_i$ if $i < j$.

Thinking about X_n as being the set of n-simplices and d_i the face map mapping to the $(n-1)$-simplex that elides the the ith vertex makes the connection between Delta sets and simplicial complexes more clear. The last condition corresponds to an emergent property of face maps of simplicial complexes: if you remove a vertex from a simplex to get one of its faces, the other vertices afterwards "shift over one," which then leads to this identity.

Note that there are Delta sets that are not simplicial complexes! Consider the Delta set where $X_0 = \{[0], [2]\}, X_1 = \{[0, 1], [0, 2]\}, X_2 = \{[0, 1, 2]\}$. This is not a simplicial complex as there are not all possible subsets of the element of X_2, but it is a Delta set, defining the face maps in the way one would expect from simplicial complexes (d_i eliding the ith vertex). We can in fact think of this space as a cone, obtained from taking a 2-simplex where points [0] and [1] have been identified by "gluing the edge [0,2] to the edge [1,2]"[2].

The last concept we need before we can introduce simplicial sets in earnest is the notion of a **degenerate simplex**. A degenerate simplex is a simplex where at least one of the vertices has been repeated. This way we can, for example, view any p-simplex as a $p+1$-simplex where one of vertices has been repeated, e.g considering the 1-simplex [0,1] as the degenerate 2-simplex [0,1,1] (or the degenerate 2-simplex [0,0,1]). Much as the idea of a face of a simplex leads to a face map, this in turn leads to **degeneracy maps**. The degeneracy map s_i maps map a p-simplex to a degenerate $p+1$-simplex obtained by repeating the i-th vertex, e.g such that $s_1([0, 1, 2]) = [0, 1, 1, 2]$. From this definition, we have the tools we need to finally define simplicial sets.

Definition 2.4. A **Simplicial set** is a sequence of sets $X_0, X_1, \ldots$ such that for each $n \geq 0$, we have maps $d_i : X_{n+1} \to X_n$ and maps $s_i : X_n \to X_{n+1}$ for $0 \leq i \leq n+1$ such that these maps obey the following relations:

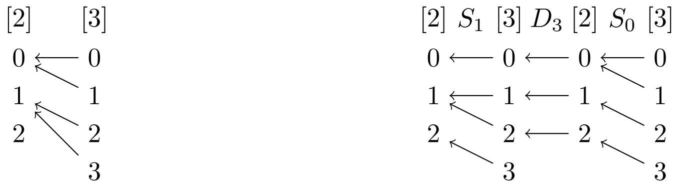
- if $i < j$, then $d_i d_j = d_{j-1} d_i$
- if $i < j$, then $d_i s_j = s_{j-1} d_i$
- $d_j s_j = d_{j+1} s_j = \mathrm{id}$
- if $i > j+1$ then $d_i s_j = s_j d_{i-1}$
- if $i \leq j$ then $s_i s_j = s_{j+1} s_i$

These relations may seem somewhat arbitrary at first, but they are simply ones that one would expect to hold for face/degeneracy maps of simplicial complexes as we have described them. An interested reader can go back in our definitions and see why these must hold. In any case, these relations will be encoded in the categorical definition we will see next.

Most work does not take this definition of simplicial set, but rather the one that is category-theoretic. We will quickly work up to this definition.

Definition 2.5. The category Δ has as objects finite ordered sets $[n] = \{0, 1, \cdots, n\}$ and as morphisms order-preserving functions $[m] \to [n]$.

We can take the objects of this category to be (ordered) simplicial complexes. Since the morphisms are merely order-preserving rather than strictly order-preserving, this allows for degenerate maps. Define $D_i : [n] \to [n+1]$ and $S_i : [n+1] \to [n]$ such that $D_i[0, \cdots, i, \cdots, n] = [0, \cdots, n, n+1]$, and $S_i[0, \cdots, i, \cdots, n, n+1] = [0, \cdots, i, i, \cdots, n]$ for $0 \leq i \leq n$. All of the morphisms in Δ are generated by these maps [2]. These are known as the coface maps and the codegeneracy maps respectively. As an example of how these maps can generate all of the morphisms in Δ, consider the order-preserving map $f : [3] \to [2]$ corresponding Figure 1a.



(a) An order preserving function $f : [3] \to [2]$

(b) f as $S_1 \circ D_3 \circ S_0$

Figure 1: A demonstration of the fact that S_i and D_i generate the morphisms in Δ.

We can then see (in Figure 1b) that f is the composition of the generating coface and codegeneracy maps S_1, D_3, and S_0. The opposite map of $D_i : [n] \to [n+1]$, which we will notate $d_i : [n+1] \to [n]$, corresponds directly to face maps in the set-theoretic definition, e.g such that $d_1([0,1,2]) = [0,2]$. Likewise, the opposite map of $S_i : [n+1] \to [n]$ is notated $s_i : [n] \to [n+1]$ and corresponds to the degeneracy maps in the set-theoretic definition. Dually, these maps generate the morphisms in Δ^{op}.

This gives us the maps and objects that seem like what we want, but how do we construct a particular simplicial set? The answer lies in thinking about a simplicial set as the (functorial) image of (standard) simplices.

Definition 2.6. A **simplicial set** is a contravariant functor $X : \Delta \to \mathbf{Set}$, i.e a covariant functor $X : \Delta^{op} \to \mathbf{Set}$.

This ends up lining up very well with our above definition of simplicial sets. We have an element $[n] \in \Delta$ for every standard n-simplex, and view $X([n])$ as the set of n-simplices (the X_ns from definition 2.4). Similarly, the functions $X(d_i) : X_n \to X_m$ and $X(s_i) : X_m \to X_n$ for $d_i : [n] \to [m]$ and $s_i : [m] \to [n]$ can the thought of as the face and degeneracy maps applied to each simplex in X_n and X_m respectively.

Much in the way that we can realize abstract simplicial complexes as geometric simplicial complexes, we can also realize simplicial sets topologically, this time as CW complexes[2]. An illustrative example is to see how the realization of a simplicial set may be an n-sphere. Consider a simplicial set with only two non-degenerate simplexes, $[0] \in X_0$ and $[0, \cdots, n]$. There are then only one (degenerate) simplex for dimension $i \neq 0, n$ — namely $[0, \cdots, 0]$. As such, the faces for the non-degenerate n-simplex must be $[0, \cdots, 0]$ (for some number of 0s), thus forcing all the faces of the n-simplex to be identified in the realization, leading to the construction of S^n by collapsing the boundary of an n-cell to a point[2]. This at the very least points to the intuition behind the result — the degenerate n-simplexes force gluing just in the way that we would expect of a CW complex.

3 The Nerve of a Category

Definition 3.1. The **nerve** of a small category $\mathcal{C}$ (written $\mathcal{N}(\mathcal{C})$) is the simplicial set whose p-simplices are diagrams in $\mathcal{C}$ of the form

$$X_0 \to X_1 \to \cdots \to X_p$$

where every X_n is an object of $\mathcal{C}$.[1]

It may not be immediately clear from this definition how this is a simplicial set. Note first that we can regard each of these diagrams as a poset (the exact details of which we will discuss shortly). Each of these posets of chains of p morphisms is isomorphic to $[p] \in \Delta$. This isomorphism allows us to speak somewhat loosely and talk about chains of morphisms as simplices.

The i-th face map is the map that takes this simplex to the simplex obtained as follows: if $i \neq 0, p$ then we delete X_i and compose the morphisms that connected it to X_{i-1} and X_{i+1}. When $i = 0$ or p we delete X_i and also the morphism incident with it. The i-th degeneracy similarly is the map that takes this simplex to the simplex obtained by replacing X_i with $X_i \to X_i$ (using the identity morphism). The geometric realization of this simplicial set is a CW complex known as the **classifying space** of $\mathcal{C}$. In this CW complex, the p-cells are in a one-to-one correspondence with the non-degenerate p-simplices of the nerve, that is to say, the simplices where none of the arrows are identity maps.

3.1 Examples

3.1.1 A Poset

Let P be a partially ordered set realized as a category, such as the poset whose Hasse diagram is depicted in figure 2. This becomes a category by saying that there is a unique morphism $x \to y$ if and only if $x \leq y$ in the poset. For this reason we have put arrows on the edges in the Hasse diagram to indicate that they are morphisms, but not all morphisms in the category are shown in this way. There are also the identity morphisms, as well as a unique morphism $p_2 \to p_5$ that equals both composites of morphisms obtained by going round the two edges of the diamond. The nerve $\mathcal{N}(P)$ is then the simplicial set with non-degenerate 0-simplices $X_0 = \{[p_1], [p_2], [p_3], [p_4], [p_5]\}$, 1-simplices $X_1 = \{[p_1, p_2], [p_2, p_3], [p_2, p_4], [p_3, p_5], [p_4, p_5]\}$ and so on with seven 2-simplices and two 3-simplices. Note that this simplicial set is also a simplicial complex. As such, the classifying space of the the category is also a (geometric) simplicial complex, although there is no guarantee of this in general.

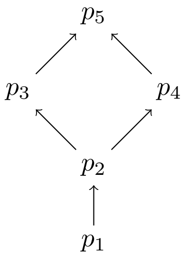


Figure 2: A poset realized as a category

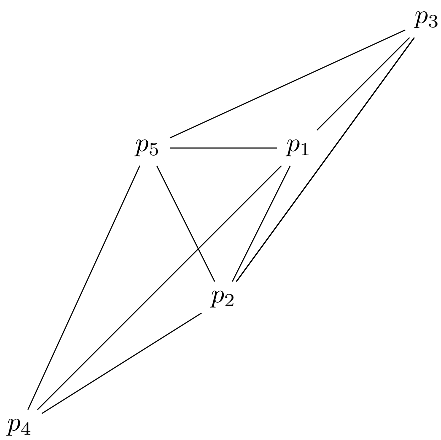


Figure 3: The geometric realization of $\mathcal{N}(P)$

4 $\mathcal{N}(C_2)$

The goal of this section is to investigate and describe the nerve for a seemingly very simple category, the cyclic group of order 2, denoted C_2, something that will prove harder than expected. When we consider C_2 as a category there is a single object $\bullet$, and we consider the elements $\{1, g\}$ of the group as morphisms, such that $g \circ g = 1$.

Figure 4: C_2 realized as a category

4.1 $\mathbb{RP}^n$

As one can glean from the next heading, real projective space of dimension n, denoted $\mathbb{RP}^n$ is going to be important for our understanding of $\mathcal{N}(C_2)$. What actually is real projective space, then? The traditional intuitive explanation is that projective space extends euclidean space with extra "points at infinity," where parallel lines meet, similar to the effect of perspective in art and optics (think of train tracks reaching the horizon). How do we represent this mathematically? There exist several homeomorphic representations. One is to define $\mathbb{RP}^n$ as the quotient $(\mathbb{R}^{n+1} - \{0\})/\sim$ such that $\forall\, 0 \neq \lambda \in \mathbb{R}.\, x \sim \lambda x$. This can be identified as the set of lines that pass through the origin, in a Euclidean space of dimension one higher. Now, imagine drawing the n-sphere S^n around all of these lines. Each line will meet the sphere exactly twice, at antipodal points. Because of this, there exists a homeomorphism between the set of lines passing through the origin and the set of pairs of antipodal points on the sphere. We represent this as the quotient $S^n/\sim$, where $\sim$ identifies antipodal points, i.e $x \sim y$ if and only if $y = -x$ or $y = x$. We see that $\mathbb{RP}^n$ admits the structure of a CW-complex, since $S^n/\sim$ is further homemorphic to the n-hemisphere with boundary, with the antipodal points only at the boundary identified. Since the boundary for the n-hemisphere is the $(n-1)$-sphere, the boundary under this quotient map is then $\mathbb{RP}^{n-1}$. We get the CW complex structure by attaching an n-cell to $\mathbb{RP}^{n-1}$ with the quotient map $q : S^{n-1} \to \mathbb{RP}^{n-1}$ as the attaching map. It follows inductively that $\mathbb{RP}^n$ is a CW complex with one cell in each dimension less than or equal to n.

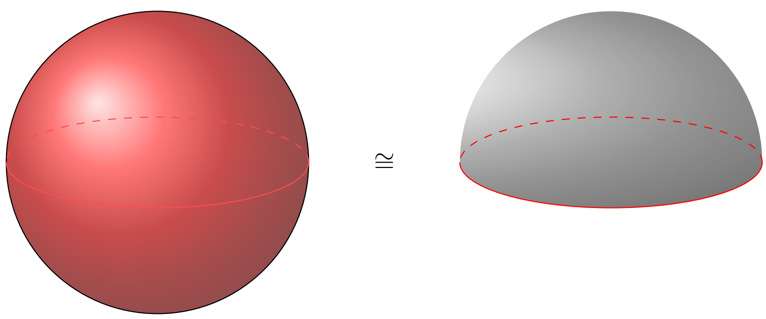


Figure 5: $\mathbb{RP}^2$ as the quotient of a sphere, and the quotient of a hemisphere. The red/pink regions are those where antipodal points are identified.

4.2 $|\mathcal{N}(C_2)| = \mathbb{RP}^\infty$

Let's (finally) determine what the (geometric realization of) the nerve of C_2. We will proceed by computing subsequent n-skeletons of $\mathcal{N}(C_2)$. First, consider the 0-skeleton, i.e chains of 0 morphisms. This is exactly the set of objects of the the category, in this case just $X_0 = \{\bullet\}$. We would represent this geometrically as a point (as you might expect). Now, consider the 1-skeleton. There are two chains of one morphism, $\bullet \xrightarrow{g} \bullet$ and $\bullet \xrightarrow{1} \bullet$. Note that $\bullet \xrightarrow{1} \bullet$ is a degenerate morphism, $(1(\bullet) = \bullet)$ and as such is an element of the 0-skeleton, so we don't need to consider it here. Consequently, we have $X_1 = \{\bullet \xrightarrow{g} \bullet\}$.

When thinking about the geometric realization of a chain, we will first consider all of the instances of $\bullet$ as distinct, before gluing them together to one point in the final representation. This will become more important in higher skeletons. Regarding g as a 1-simplex gives us a line, and the fact that it starts and ends at the same object identifies the endpoints, which gives us the 1-sphere (the circle), as pictured in figure 6. Note that $\mathbb{RP}^1$ is homeomorphic to the circle.

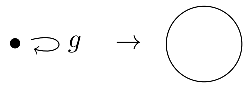


Figure 6: The geometric realization of the 1-skeleton of $\mathcal{N}(C_2)$

Consider now the 2-skeleton, which is $X_2 = \{\bullet \xrightarrow{g} \bullet \xrightarrow{g} \bullet\}$. We will label each of the different instances of $\bullet$, in this case such that $X_2 = \{a \xrightarrow{g} b \xrightarrow{g} c\}$. We then need to consider all of the faces, which are $a \xrightarrow{g} b$, $b \xrightarrow{g} c$ and $a \xrightarrow{g \circ g} c$, the last of which is equal to $a \xrightarrow{1} c$. Treating the element of X_2 as a 2-simplex with those faces gives the simplex depicted in Figure 7.

In Figure 7 the left triangle and middle graph represent 2-dimensional regions whose boundary is shown, which are topologically the same as the disc on the right. We obtain the middle graph from the left graph by collapsing the the chain $a \xrightarrow{1} c$ to one point. We then end up with a circle connected by two identified 1-simplices, and two identified points, which is homeomorphic with the closed 2-ball with antipodal points on the boundary identified. Note that the closed 2-ball with antipodal points identified on the boundary is homeomorphic to $\mathbb{RP}^2$, since the 2-ball is homeomorphic to the 2-hemisphere.

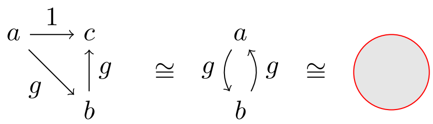


Figure 7: The geometric realization of the 2-skeleton of $\mathcal{N}(C_2)$. Red here again represents that antipodal points are identified, and the filled grey indicates that the rightmost shape is the closed 2 dimensional disk, rather than its boundary circle.

Consider now the 3-skeleton of $\mathcal{N}(C_2)$. The 3-skeleton is $X_3 = \{\bullet \xrightarrow{g} \bullet \xrightarrow{g} \bullet \xrightarrow{g} \bullet\}$, by similar reasoning as above. Similar to the 2-skeleton, we label the objects a, b, c, d. The faces are $a \xrightarrow{g} b \xrightarrow{g} c$, $b \xrightarrow{g} c \xrightarrow{g} d$, $a \xrightarrow{1} c \xrightarrow{g} d$, and $a \xrightarrow{g} b \xrightarrow{1} d$. Drawing this out yields a 3-simplex as depicted in figure 8. Collapsing all of the chains that consist of the identity morphism collapses the entire simplex to just two points and two 1-simplices that are identified, this time enclosing a three dimensional region, as depicted in figure 9.

After these identifications, we see from the diagrams that the 3-skeleton is a 3-ball $\bullet \xrightarrow{g} \bullet \xrightarrow{g} \bullet \xrightarrow{g} \bullet$ whose boundary 2-sphere is the union of 2 copies of the single 2-cell $\bullet \xrightarrow{g} \bullet \xrightarrow{g} \bullet$. These 2-cells are the faces of the 3-cell obtained by omitting the morphisms g at the two ends of the chain. These 2-cells intersect along the equator of the 3-ball, which is 2 copies of the 1-cell $\bullet \xrightarrow{g} \bullet$. This is very close to showing that we have $\mathbb{RP}^3$, since we now have a 3-ball with parts of the boundary identified, but we still need to show that this identification must be the antipodal map. Why must we identify "opposite points" of the sphere? Initially, it may appear that we could another map, say by identifying points on the 2-sphere by $(x, y, z) \sim (x, y, -z)$. However, the aforementioned 1-cells force use to choose the antipodal map. Looking at figure 9, we can see that there is a directionality at play: the beginning of the 1-cell $a \xrightarrow{g} b$ is identified with the beginning of the 1-cell $b \xrightarrow{g} a$, which is an antipodal identification (as we saw in the 2-skeleton). In order to have a continuous identification of each of the 2-cells, we have to extend the antipodal identification from the 1-cells to the 2-cells.

This pattern repeats inductively as we go into higher dimensions. For all $n \geq 2$, there are exactly two non-degenerate faces, obtained from eliding the first and and last morphism. Before any gluing, the geometric realization is an n-ball, the boundary of which is the union of these two non-degenerate $(n-1)$-cells (since all other faces are degenerate), which are identified with the antipodal map for the exact reason identified in the case of the 3-skeleton.

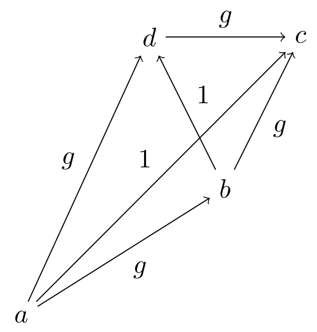


Figure 8: The 3-skeleton of $\mathcal{N}(C_2)$

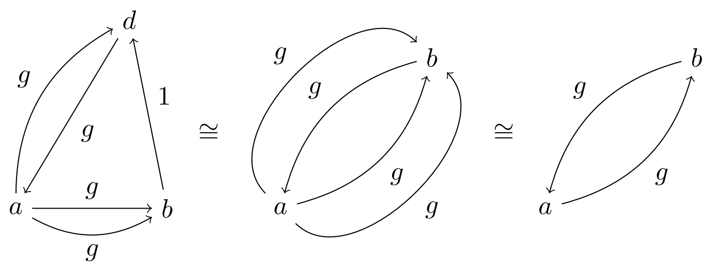


Figure 9: The 3-skeleton of $\mathcal{N}(C_2)$ after a series of gluings. First gluing along the identity morphisms, and then collapsing discs of two arrows going the same direction into single points.

References

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